

XLI. *A Method of finding, by the Help of Sir Ifaac Newton's binomial Theorem, a near Value of the very slowly-converging infinite Series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \&c.$ when x is very nearly equal to 1. By Francis Maferes, Esq. F. R. S. Cursitor Baron of the Exchequer.*

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ARTICLE I.

IF the capital letters A, B, C, D, E, &c. be put for the numeral co-efficients of the powers of x in the said series, so that A shall be = 1, $B = \frac{1}{2}$, $C = \frac{1}{3}$, $D = \frac{1}{4}$, $E = \frac{1}{5}$, and so on, we shall have $B = \frac{1}{2} \times A$, $C = \frac{2}{3} \times B$, $D = \frac{3}{4} \times C$, $E = \frac{4}{5} \times D$, &c. and the series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \&c.$ will be $= x + \frac{1}{2} A x x + \frac{2}{3} B x^3 + \frac{3}{4} C x^4 + \frac{4}{5} D x^5 + \frac{5}{6} E x^6 + \frac{6}{7} F x^7 + \frac{7}{8} G x^8 + \&c.$; in which series the fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, &c. which generate the co-efficients of the powers of x in the several terms after the first term x , are derived from each other by the continual addition of 1 to both their numerators and their denominators.

2. This

2. This observation suggests to us a method of finding a near value of the sum of this series by the help of Sir ISAAC NEWTON's binomial theorem, which may be explained as follows.

If m and n represent any two whole numbers, the reciprocal of the $\frac{m}{n}$ -th power of the binomial quantity $1-x$, or, according to Sir ISAAC NEWTON's notation of powers, the quantity $\overline{1-x}^{-\frac{m}{n}}$, will, according to that celebrated theorem, be equal to the infinite series

$$\begin{aligned} & 1 + \frac{m}{n} \times x \\ & + \frac{m}{n} \times \frac{m+n}{2n} \times xx + \frac{m}{n} \times \frac{m+n}{2n} \times \frac{m+2n}{3n} \times x^3 \\ & + \frac{m}{n} \times \frac{m+n}{2n} \times \frac{m+2n}{3n} \times \frac{m+3n}{4n} \times x^4 \\ & + \frac{m}{n} + \frac{m+n}{2n} \times \frac{m+2n}{3n} \times \frac{m+3n}{4n} \times \frac{m+4n}{5n} \times x^5 + \&c. \end{aligned}$$

$$\text{OR } 1 + \frac{m}{n} A x + \frac{m+n}{2n} B x x + \frac{m+2n}{3n} C x^3 + \frac{m+3n}{4n} D x^4 + \frac{m+4n}{5n} E x^5 \\ + \frac{m+5n}{6n} F x^6 + \frac{m+6n}{7n} G x^7 + \frac{m+7n}{8n} H x^8 + \&c.; \text{ in which series}$$

the capital letters A, B, C, D, E, F, G, H, &c. stand for 1 and the co-efficients of x , xx , x^3 , x^4 , x^5 , x^6 , x^7 , x^8 , &c.

Now it is evident, that the generating fractions $\frac{m+n}{2n}$, $\frac{m+2n}{3n}$, $\frac{m+3n}{4n}$, $\frac{m+4n}{5n}$, $\frac{m+5n}{6n}$, $\frac{m+6n}{7n}$, $\frac{m+7n}{8n}$, &c. are derived from $\frac{m}{n}$ and from each other by the continual addition of n to both their numerators and denominators. Therefore, though they are greater than they would be if m was subtracted from the numerator

of

infinite Series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \&c.$ 897

of each of them, that is, than the fractions $\frac{n}{2n}, \frac{2n}{3n}, \frac{3n}{4n}, \frac{4n}{5n}, \frac{5n}{6n}, \frac{6n}{7n}, \frac{7n}{8n}, \&c.$ and consequently, than the fractions $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \&c.$ which are respectively equal to $\frac{n}{2n}, \frac{2n}{3n}, \frac{3n}{4n}, \frac{4n}{5n}, \frac{5n}{6n}, \frac{6n}{7n}, \frac{7n}{8n}, \&c.$ Yet, the further we go in the series, the less is the proportion in which they exceed the latter fractions; infomuch that, if we go far enough in the series, we may find terms in it whose proportion to the corresponding terms in the series $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \&c.$ shall approach as near as we please to a proportion of equality. And, by taking n of a very great magnitude in comparison of m , we may even make the first terms of the series $\frac{m+n}{2n}, \frac{m+2n}{3n}, \frac{m+3n}{4n}, \frac{m+4n}{5n}, \frac{m+5n}{6n}, \frac{m+6n}{7n}, \frac{m+7n}{8n}, \&c.$ approach very nearly to an equality with the corresponding terms of the series $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \&c.$ which are the generating fractions of the proposed series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \&c.$ In order to this, let m be taken = 1, and $n = 1,000,000,000,000$, that is, = a billion, or the square of a million, which, to avoid the frequent repetition of so many cyphers, we will call b .

Then will $\frac{1}{1-x} = \frac{1}{1,000,000,000,000}$, or $\frac{1}{1-x} = \frac{1}{b}$, or $\frac{1}{1-x} = \frac{1}{b}$, be

$$= 1 + \frac{1}{b} A x + \frac{1+b}{2b} B x^2 + \frac{1+2b}{3b} C x^3 + \frac{1+3b}{4b} D x^4 + \frac{1+4b}{5b} E x^5 + \frac{1+5b}{6b} F x^6 + \frac{1+6b}{7b} G x^7 + \frac{1+7b}{8b} H x^8 + \&c. \text{ which, on account of}$$

the great magnitude of b , $2b$, $3b$, $4b$, $5b$, $6b$, $7b$, &c. in comparison of 1, will be almost equal to (though somewhat greater than) $1 + \frac{1}{b} A x + \frac{b}{2b} B x x + \frac{2b}{3b} C x^3 + \frac{3b}{4b} D x^4 + \frac{4b}{5b} E x^5$

$+ \frac{5b}{6b} F x^6 + \frac{6b}{7b} G x^7 + \frac{7b}{8b} H x^8 + \&c.$ or $1 + \frac{1}{b} A x + \frac{1}{2} B x x + \frac{2}{3} C x^3 +$

$\frac{3}{4} D x^4 + \frac{4}{5} E x^5 + \frac{5}{6} F x^6 + \frac{6}{7} G x^7 + \frac{7}{8} H x^8 + \&c.$ or $1 + \frac{1}{b} \times 1 \times x$

$+ \frac{1}{2} \times \frac{1}{b} \times 1 \times x x + \frac{2}{3} \times \frac{1}{2} \times \frac{1}{b} \times 1 \times x^3 + \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{b} \times 1 \times x^4 + \&c.$

or $1 + \frac{1}{b} x + \frac{1}{b} \times \frac{1}{2} x x + \frac{1}{b} \times \frac{1}{3} x^3 + \frac{1}{b} \times \frac{1}{4} x^4 + \&c.$ or $1 + \frac{x}{b} + \frac{x^2}{2b}$

$+ \frac{x^3}{3b} + \frac{x^4}{4b} + \frac{x^5}{5b} + \frac{x^6}{6b} + \frac{x^7}{7b} + \frac{x^8}{8b} + \&c.$ Therefore, multiplying

both sides by b , we shall have $b \times \frac{1}{1-x} \frac{1}{b}$ nearly $= b + x + \frac{x^2}{2}$

$+ \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \&c.$; and, subtracting b from

both sides, $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \&c.$ nearly

$= b \times \frac{1}{1-x} \frac{1}{b} - b = b \times \left[\frac{1}{1-x} \right] \frac{1}{b} - b$; that is, the proposed series

$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \&c.$ will be nearly equal to

$b \times \left[\frac{1}{1-x} \right] \frac{1}{b} - b$. We must therefore first subtract x from

1, and then divide 1 by the remainder, which will give

us a quotient equal to $\frac{1}{1-x}$. And, having found

this quotient, we must extract its b th, or

1,000,000,000,000th, root, and multiply the said root

by b , or 1,000,000,000,000; and, lastly, from the pro-

duct we must subtract b , or 1,000,000,000,000: and

the

infinite Series $x + \frac{xx}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \&c.$ 899

the remainder thereby obtained will be nearly equal to the proposed infinite series $x + \frac{xx}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \&c.$ Q. E. I.

An example of the foregoing method of summing the said infinite series.

3. As an example of this method of finding the value of the series $x + \frac{xx}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \&c.$ let us suppose x to be equal to $\frac{9}{10}$.

Then we shall have $1 - x = 1 - \frac{9}{10} = \frac{1}{10}$, and $\frac{1}{1-x} = 10$. Now, since the logarithm of 10 in BRIGGS's System of logarithms is 1, the logarithm of the 1,000,000,000,000th root of 10 must be the 1,000,000,000,000th part of 1, or must be $= .000,000,000,001$. This logarithm is too small to be found in the common tables of logarithms, which go only to seven places of figures; and therefore the number corresponding to it, that is, the 1,000,000,000,000th root of 10, cannot be found by the help of those tables; but it may be found in the manner following. The 1,000,000,000,000th root of 10 is a number that is somewhat, and but a very little, greater than 1. That number, therefore, and 1 will represent two ordinates to the axis, or asymptote, of a logarithmick curve that are very nearly contiguous to each other: whence it follows, that the sub-tangent of the curve will

bear very nearly the same proportion to the lesser ordinate x , as the absciss of the axis intercepted between the two ordinates, that is, as the logarithm of the ratio of the greater ordinate to the lesser, or the logarithm .000,000,000,001, bears to the difference of the said ordinates. Say therefore, as .434,294,481,9 (which is the sub-tangent of the logarithmick curve in BRIGGS's System of Logarithms) is to 1 (or the lesser of the two ordinates) so is .000,000,000,001 to a fourth number, which will be .000,000,000,002,302,585,093; and this fourth number will be the excess of the greater of the said two ordinates above the lesser, or of the billionth root of 10 above 1. Therefore the billionth root of 10 will be = 1.000,000,000,002,302,585,093; which, being multiplied by 1,000,000,000,000, will be = 1,000,000,000,002.302,585,093; from which if we subtract 1,000,000,000,000, the remainder will be 2.302,585,093. Therefore 2.302,585,093 is nearly equal to the infinite series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + 8c.$ when x is = $\frac{9}{10}$. Q. E. I.

4. This number 2.302,585,093 gives the value of the series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + 8c.$ exact to nine places of figures, the error being only in the 10th figure 3, which ought to be a 2 instead of a 3, the more accurate value of that series (which is equal to the logarithm of

infinite Series $x + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{4} + \frac{x^5}{6} + \&c.$ 901

the ratio of 1 to $1-x$ in NAPIER's System of Logarithms, that is, in the present example, to the logarithm of the ratio of 1 to $\frac{1}{10}$, or of 10 to 1, or to NAPIER's logarithm of 10) being 2.302,585,092,994,04.

5. I believe that similar applications of the binomial theorem may be made for the summation of other slowly-converging infinite serieses, whenever the generating fractions (by the multiplication of which the numeral co-efficients of the terms of such serieses are produced from each other) are formed by the addition of a given number to both their numerators and denominators. In our endeavours, therefore, to sum such serieses it will be proper to attend to the law of the said generating fractions.

